

## REAL NUMBERS THAT HAVE GOOD DIOPHANTINE APPROXIMATIONS OF THE FORM $r_{n+1}/r_n$

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**Abstract.** In this note, we show that if  $\alpha$  is a real number such that there exist a constant  $c$  and a sequence of non-zero integers  $(r_n)_{n \geq 0}$  with  $\lim_{n \rightarrow \infty} |r_n| = \infty$  for which  $\left| \alpha - \frac{r_{n+1}}{r_n} \right| < \frac{c}{|r_n|^2}$  holds for all  $n \geq 0$ , then either  $\alpha \in \mathbb{Z} \setminus \{0, \pm 1\}$  or  $\alpha$  is a quadratic unit. Our result complements results obtained by P. Kiss who established the converse in *Period. Math. Hungar.* 11 (1980), 281–287.

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### 1. Introduction

Let  $\alpha$  be a real number. In this paper, we deal with the topic of approximating  $\alpha$  by rationals. It is well known that there exist a constant  $c$  and two sequences of integers  $(u_n)_{n \geq 0}$  and  $(v_n)_{n \geq 0}$  with  $v_n > 0$  for all  $n \geq 0$  and  $v_n$  diverging to infinity (with  $n$ ) such that

$$(1) \quad \left| \alpha - \frac{u_n}{v_n} \right| \leq \frac{c}{v_n^2}$$

holds for all  $n \geq 0$ . By work of Hurwitz (see [5]), one can take  $c := 1/\sqrt{5}$  and the above constant is well known to be best-possible for  $\alpha := \frac{1 + \sqrt{5}}{2}$ .

Several papers in the literature deal with the question of approximating  $\alpha$  by rationals  $u_n/v_n$  requiring  $u_n$  and  $v_n$  to satisfy (1) as well as some additional conditions. For example, if  $\alpha$  is irrational and  $a, b$  and  $k$  are integers with  $k > 1$ , then there exist a constant  $c$  and two sequences of integers  $(u_n)_{n \geq 0}$  and  $(v_n)_{n \geq 0}$  with  $v_n > 0$  and  $v_n$  diverging to infinity such that

$$(2) \quad \left| \alpha - \frac{u_n}{v_n} \right| < \frac{c}{v_n^2} \quad \text{and} \quad u_n \equiv a \pmod{k}, \quad v_n \equiv b \pmod{k}$$

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holds for all  $n \geq 0$ . The best-possible constant  $c$  in (2) is  $k^2/4$  in case  $a$  and  $b$  are not both divisible by  $k$  (see [3] and [4]).

If  $\alpha$  is algebraic and  $\mathcal{P}$  is a fixed finite set of prime numbers, then Ridout [10] inferred from Roth's work [11] that one cannot approximate  $\alpha$  too well by rational numbers  $u/v$  where either  $u$  or  $v$  is divisible only by primes from  $\mathcal{P}$ . More precisely, for every given  $\epsilon > 0$ , the inequality

$$(3) \quad \left| \alpha - \frac{u}{v} \right| < \frac{1}{v^{1+\epsilon}}$$

has only finitely many integer solutions  $(u, v)$  with  $v > 0$  and either  $u$  or  $v$  divisible by primes from  $\mathcal{P}$ , only.

A different type of question was considered by P. Kiss in [6] and [7] (see also [8] and [9]). In [6], it was shown that if  $\alpha$  is a quadratic unit with  $|\alpha| > 1$ , then there exist a constant  $c$  and a sequence of integers  $(r_n)_{n \geq 0}$  with  $|r_n|$  diverging to infinity such that

$$(4) \quad \left| \alpha - \frac{r_{n+1}}{r_n} \right| < \frac{c}{|r_n|^2}$$

holds for all  $n \geq 0$ . In [7] it was shown that, in fact, a statement similar to (4) holds for both  $\alpha$  and  $\alpha^s$  where  $s \geq 2$  is some positive integer: There exist a constant  $c$  and a sequence of integers  $(r_n)_{n \geq 0}$  with  $|r_n|$  diverging to infinity such that both

$$(5) \quad \left| \alpha - \frac{r_{n+1}}{r_n} \right| < \frac{c}{|r_n|^2} \quad \text{and} \quad \left| \alpha^s - \frac{r_{n+s}}{r_n} \right| < \frac{c}{|r_n|^2}$$

hold for all  $n \geq 0$ .

An explicit description of a sequence  $(r_n)_{n \geq 0}$  satisfying inequalities (5) above was also given in [7]: Let

$$f = X^2 + AX + B \quad (A, B \in \mathbf{Z})$$

be the minimal polynomial of  $\alpha$  over  $\mathbf{Q}$ . Let  $\beta$  be the other root of  $f$ . Since  $\alpha$  is a unit,  $|B| = |\alpha\beta| = 1$  must hold which implies that the sequence

$$(6) \quad r_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1$$

fulfills the inequalities (5) for all  $n$  with  $c := 2 \sum_{i=0}^{s-1} |\alpha|^i |\beta|^{s-1-i}$ .

One may ask if one can characterize all real numbers  $\alpha$  for which there exist a constant  $c$  and a sequence of integers  $(r_n)_{n \geq 0}$  with  $|r_n|$  diverging to infinity such that inequality (4) or, respectively, inequalities (5) hold for all  $n \geq 0$ . From the above remarks, we saw that quadratic units  $\alpha$  with  $|\alpha| > 1$  have these properties. Moreover, the sequence  $r_n := \alpha^n$  ( $n \geq 1$ ) shows that integers  $\alpha$  with  $|\alpha| > 1$  also belong to this class. It seems natural therefore to inquire if there are any other candidates  $\alpha$  satisfying the above conditions. The perhaps not too surprising, answer is no. Our exact result is the following.

**Theorem 1.** *Let  $\alpha$  be a real number.*

(i) *Assume that there exist  $\epsilon > 0$  and a sequence of integers  $(r_n)_{n \geq 0}$  with  $|r_n|$  diverging to infinity such that*

$$(7) \quad \left| \alpha - \frac{r_{n+1}}{r_n} \right| < \frac{1}{|r_n|^{\frac{3}{2} + \epsilon}}$$

*holds for all  $n \geq 0$ . Then,  $\alpha$  is a real algebraic integer of absolute value larger than 1 and of degree at most 2. Moreover, if  $\alpha$  is irrational, then the absolute value of its norm is smaller than  $\sqrt{|\alpha|}$ .*

(ii) *Assume, moreover, that there exist a constant  $c$  and a sequence of integers  $(r_n)_{n \geq 0}$  with  $|r_n|$  diverging to infinity such that*

$$(8) \quad \left| \alpha - \frac{r_{n+1}}{r_n} \right| < \frac{c}{|r_n|^2}$$

*holds for all  $n \geq 0$ . Then  $\alpha$  is a quadratic unit or a rational integer different from 0 or  $\pm 1$ .*

The following result characterizes real numbers  $\alpha$  for which - as in (5) - two different powers can be well approximated by rationals.

**Theorem 2.** *Let  $\alpha$  be a real number. Assume that there exist two coprime positive integers  $s_1$  and  $s_2$ , two positive integers  $t_1$  and  $t_2$ , a real number  $\epsilon > 0$ , and a sequence of integers  $(r_n)_{n \geq 0}$  with  $|r_n|$  diverging to infinity with  $n$  such that*

$$(9) \quad \left| \alpha^{s_i} - \frac{r_{n+t_i}}{r_n} \right| < \frac{1}{|r_n|^{\frac{3}{2} + \epsilon}}$$

*hold for all  $n \geq 0$  and for both  $i = 1$  and  $2$ . Then, either  $\alpha \in \mathbf{Z} \setminus \{0, \pm 1\}$  or  $\alpha$  is quadratic irrational with norm smaller than  $\sqrt{|\alpha|}$  in absolute value. If moreover  $\alpha$  is irrational and there exists a constant  $c$  with*

$$(10) \quad \left| \alpha^{s_1} - \frac{r_{n+t_1}}{r_n} \right| < \frac{c}{r_n^2},$$

*then  $\alpha$  is a quadratic unit.*

The proofs of both Theorems 1 and 2 are based on the following result which follows right away from our recent work [1] and [2].

**Theorem DL.** *Let  $(r_n)_{n \geq 0}$  be a sequence of integers with  $|r_n|$  diverging to infinity.*

(i) *Assume that*

$$(11) \quad \overline{\lim}_{n \rightarrow \infty} \frac{|r_n^2 - r_{n+1}r_{n-1}|}{\sqrt{|r_n|}} < \frac{1}{\sqrt{2}}.$$

Then, the sequence  $\left(\frac{r_{n+1}}{r_n}\right)_{n \geq 0}$  is convergent to a limit  $\alpha$  that is a non-zero algebraic integer of degree at most 2. If  $\alpha$  is irrational, then its norm is smaller than  $\sqrt{|\alpha|}$ . Moreover, there exists  $n_0 \in \mathbb{N}$  such that  $(r_n)_{n \geq n_0}$  is binary recurrent.

(ii) If

$$(12) \quad |r_n^2 - r_{n+1}r_{n-1}| < c$$

holds for some constant  $c$  and all  $n$ , then  $\alpha$  is a quadratic unit or a non-zero integer.

We point out that in our work [1] and [2], we gave more precise descriptions for both the sequences  $(r_n)_{n \geq 0}$  satisfying (11) or (12), respectively, and the limit  $\alpha = \lim_{n \rightarrow \infty} \frac{r_{n+1}}{r_n}$ , but the above Theorem DL suffices for our present purposes.

We now proceed to the proofs of Theorems 1 and 2.

## 2. The Proofs

**Proof of Theorem 1.** We will prove (i) in detail and we will only sketch the proof of (ii).

(i) By replacing the sequence  $(r_n)_n$  by the sequence  $((-1)^n r_n)_n$  and  $\alpha$  by  $-\alpha$  if  $\alpha < 0$ , we may assume  $\alpha \geq 0$  and  $r_n > 0$  for all  $n \geq 0$ . By letting  $n$  tend to infinity in (7), we get  $\alpha = \lim_{n \rightarrow \infty} \frac{r_{n+1}}{r_n}$ . Since  $r_n$  diverges to infinity, we must have  $\alpha \geq 1$ . We now show that  $\alpha > 1$ . Indeed, if  $\alpha = 1$ , then inequality (7) becomes

$$\left|1 - \frac{r_{n+1}}{r_n}\right| < \frac{1}{r_n^{\frac{3}{2} + \epsilon}},$$

or

$$|r_{n+1} - r_n| < \frac{1}{r_n^{\frac{1}{2} + \epsilon}} \leq 1,$$

therefore  $r_{n+1} = r_n$  for all  $n \geq 0$ . This contradicts the fact that  $r_n$  diverges to infinity. Hence,  $\alpha > 1$ .

Now let  $\delta$  be a real number with  $1 < \delta < \alpha$ , note that  $\gamma := 2\alpha - \delta$  exceeds  $\alpha$ , and choose  $n_0$  such that

$$r_n^{\frac{3}{2} + \epsilon} > \frac{1}{\alpha - \delta}$$

holds for all  $n \geq n_0$ . From inequality (7), we get that

$$(13) \quad \delta r_n < r_{n+1} < \gamma r_n$$

holds for all  $n \geq n_0$ . From inequalities (7) for  $n$  and  $n+1$  and the triangular inequality, we get

$$\frac{|r_{n+1}^2 - r_n r_{n+2}|}{r_n r_{n+1}} = \left| \frac{r_{n+1}}{r_n} - \frac{r_{n+2}}{r_{n+1}} \right| < \left| \alpha - \frac{r_{n+1}}{r_n} \right| + \left| \alpha - \frac{r_{n+2}}{r_{n+1}} \right| < \left( \frac{1}{r_n^{\frac{3}{2} + \epsilon}} + \frac{1}{r_{n+1}^{\frac{3}{2} + \epsilon}} \right),$$

or

$$(14) \quad \frac{|r_{n+1}^2 - r_{n+2} r_n|}{\sqrt{r_{n+1}}} < \frac{1}{r_n^\epsilon} \cdot \sqrt{\frac{r_{n+1}}{r_n}} + \frac{1}{r_{n+1}^\epsilon} \cdot \left( \frac{r_n}{r_{n+1}} \right).$$

Using inequality (13) in (14), we get

$$(15) \quad \frac{|r_{n+1}^2 - r_{n+2} r_n|}{\sqrt{r_{n+1}}} < \frac{c_1}{r_n^\epsilon} + \frac{c_2}{r_{n+1}^\epsilon}$$

for all  $n \geq n_0$ , where  $c_1 = \sqrt{\gamma}$  and  $c_2 = 1/\delta$ . We now let  $n$  tend to infinity in (15) and get

$$(16) \quad \lim_{n \rightarrow \infty} \frac{|r_n^2 - r_{n+1} r_{n-1}|}{\sqrt{r_n}} = 0 < \frac{1}{\sqrt{2}}.$$

Consequently, the conclusion of part (i) of Theorem 1 follows from part (i) of Theorem DL.

The remaining assertions of part (ii) now follow from putting  $\epsilon := 1/2$  in (15) and invoking  $r_{n+1}/r_n < \gamma$  as well as part (ii) of Theorem DL.

Theorem 1 is therefore established.

**Remark 1.** The occurrence of  $\epsilon > 0$  in the exponent in inequality (7) is unnecessary. A closer investigation of the arguments used in the proof of Theorem 1 shows that the conclusion of part (i) of Theorem 1 remains valid if inequality (7) is replaced by the weaker inequality

$$(7') \quad \left| \alpha - \frac{r_{n+1}}{r_n} \right| < \frac{1 - \epsilon}{\sqrt{2}(\sqrt{|\alpha|} + 1/|\alpha|)} \cdot \frac{1}{r_n^{\frac{3}{2}}}.$$

**Remark 2.** Assume that  $\alpha$  is a real number such that the hypotheses of either part (i) or part (ii) of Theorem 1 are fulfilled. Using the full strength of our results from [1] and [2], we can infer that if  $\alpha$  is an integer, then  $(r_n)_{n \geq 0}$  is a geometrical progression of ratio  $\alpha$  from some  $n$  on. However, if  $\alpha$  is quadratic and the hypotheses of part (ii) of Theorem 1 are fulfilled, we can only infer that  $(r_n)_{n \geq 0}$  is binary recurrent from some  $n$  on, and that its characteristic equation is precisely the minimal polynomial of  $\alpha$  over  $\mathbf{Q}$ . However, we cannot infer that  $(r_n)_{n \geq 0}$  is the

Lucas sequence of the first kind for  $\alpha$  given by formula (6), mostly because the constant  $c$  appearing in inequality (8) is arbitrary. Of course, if one imposes that the constant  $c$  appearing in inequality (8) is small enough (for example,  $c = 1/2$ ), then the rational numbers  $r_{n+1}/r_n$  are exactly the convergents of  $\alpha$ , therefore  $r_n$  is indeed given by formula (6) for all  $n$  (up to some linear shift in the index  $n$ ).

**Proof of Theorem 2.** If one replaces the sequence  $(r_n)_{n \geq 0}$  by the sequence  $(R_n)_{n \geq 0} = (r_{nt_1})_{n \geq 0}$ , then the first inequality (9) together with part (i) of Theorem 1 show that  $\alpha^{s_1}$  is an algebraic integer, different than 0 or  $\pm 1$ , of degree at most 2. Similarly, if one replaces the sequence  $(r_n)_{n \geq 0}$  by the sequence  $(R_n)_{n \geq 0} = (r_{nt_2})_{n \geq 0}$ , then the second part of inequality (9) together with part (ii) of Theorem 1 show that  $\alpha^{s_2}$  is an algebraic integer, different than 0 or  $\pm 1$ , of degree at most 2.

From here on, all we need to establish is that  $\alpha$  is itself algebraic of degree at most 2. Assume that this is not so and let  $K := \mathbf{Q}[\alpha]$  and  $K_i := \mathbf{Q}[\alpha^{s_i}]$  for  $i = 1, 2$ . Since  $s_1$  and  $s_2$  are coprime, we get that  $K = \mathbf{Q}[\alpha^{s_1}, \alpha^{s_2}]$ . Moreover, we must have  $[K_i : \mathbf{Q}] = 2$  for both  $i = 1$  and  $2$ , i.e.  $K$  is a biquadratic real extension of  $\mathbf{Q}$  and  $\text{Gal}(K/\mathbf{Q}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ . Hence, there exist two non-trivial elements  $\sigma_1$  and  $\sigma_2$  in  $\text{Gal}(K/\mathbf{Q})$  with  $\sigma_i(\alpha^{s_i}) = \alpha^{s_i}$ , i.e.

$$(17) \quad 1 = \frac{\sigma_i(\alpha^{s_i})}{\alpha^{s_i}} = \left( \frac{\sigma_i(\alpha)}{\alpha} \right)^{s_i}$$

for  $i = 1, 2$ . Since  $K$  is a real field and  $\sigma_i$  is non-trivial, formula (17) implies that  $\sigma_i(\alpha) = -\alpha$  for  $i = 1, 2$ . Hence,  $\sigma_1(\alpha) = \sigma_2(\alpha)$ , which implies  $\sigma_1 = \sigma_2$ . But this is a contradiction. The remaining of the assertions of Theorem 2 follow from Theorem 1.

Theorem 2 is therefore established.

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